Symmetry-based determination of space-time functions in nonequilibrium growth processes

Andreas Röthlein,[1] Florian Baumann,[1,2] and Michel Pleimling[1,3]

[1]*Institut für Theoretische Physik I, Universität Erlangen-Nürnberg, D-91058 Erlangen, Germany*
[2]*Laboratoire de Physique des Matériaux (CNRS UMR 7556), Université Henri Poincaré Nancy I,*
B.P. 239, F-54506 Vandœuvre lès Nancy Cedex, France
[3]*Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0435, USA*

We study the space-time correlation and response functions in nonequilibrium growth processes described by linear stochastic Langevin equations. Exploiting exclusively the existence of space- and time-dependent symmetries of the noiseless part of these equations, we derive expressions for the universal scaling functions of two-time quantities which are found to agree with the exact expressions obtained from the stochastic equations of motion. The usefulness of the space-time functions is illustrated through the investigation of two atomistic growth models, the Family model and the restricted Family model, which are shown to belong to a unique universality class in $1+1$ and $2+1$ space dimensions. This corrects earlier studies which claimed that in $2+1$ dimensions the two models belong to different universality classes.

I. INTRODUCTION

Fluctuations are omnipresent when analyzing surfaces and interfaces. These fluctuations can be equilibrium fluctuations, as encountered, for example, when looking at steps on surfaces, or they can be of nonequilibrium origin as is the case in various growth processes. Well-known examples of non-equilibrium surface fluctuations are found in kinetic roughening or nonequilibrium growth processes [1–3] such as, for example, in thin film growth due to vapor deposition. Interestingly, both equilibrium and nonequilibrium interface and surface fluctuations can be described on a mesoscopic level through rather simple Langevin equations [4]. In this approach the fast degrees of freedom are modeled by a noise term, thus yielding stochastic equations of motion for the slow degrees of freedom. In many instances the physics of dynamical processes is to a large extend captured by linearized Langevin equations [5] where one distinguishes whether the dynamics is purely diffusive or whether mass conservation has to be implemented.

For purely diffusive dynamics (called model A dynamics in critical dynamics [6,7]) the linear Langevin equation can be written in the following way:

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nu_2 \nabla^2 h(\mathbf{x},t) + \lambda + \eta(\mathbf{x},t), \tag{1}$$

where $h(\mathbf{x},t)$ is the value of the macroscopic field h at site $\mathbf{x}$ at time t. In the physical context of fluctuating interfaces and growth processes in $d+1$ spatial dimensions h is the height field whereas $\mathbf{x}$ is the lateral position in the underlying d-dimensional substrate lattice. In addition, $\nu_2 > 0$ is the diffusion constant whereas λ is the mean growth velocity (which may of course be zero). Finally, the random variable η models the noise due to the fast degrees of freedom. Depending on the physical problem at hand, either Gaussian white noise or spatially and/or temporally correlated noise is usually considered [8].

In the context of kinetic roughening and nonequilibrium growth processes Eq. (1) is called the Edwards-Wilkinson

(EW) equation [9]. This equation has been used for the description of many dynamical processes, such as, for example, equilibrium step fluctuations with random attachment/detachment events at the step edge [10–12]. This equation also describes the dynamics of a growing surface with a normal incidence of the incoming particles. An obliquely incident particle beam, however, generates anisotropies which can only be described by a more complex nonlinear Langevin equation [13].

In growth processes with mass conservation the following linear Langevin equation (with $\nu_4 > 0$)

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\nu_4 \nabla^4 h(\mathbf{x},t) + \lambda + \eta(\mathbf{x},t) \tag{2}$$

has been proposed [14,15]. This equation is sometimes called the noisy Mullins-Herring (MH) equation. The noise term again reflects the physics of the investigated system. In the case of equilibrium fluctuations conserved noise must be considered, leading to the so-called model B dynamics [6,7]. On the other hand, when studying out-of-equilibrium processes one can again focus on Gaussian white noise or on noise which is correlated in space and/or time [16].

The Langevin equation (2) is used, for example, to describe film growth via molecular beam epitaxy [15,17,18], equilibrium fluctuations limited by step edge diffusion [12,19] or even tumor growth [20,21].

An important notion in nonequilibrium growth processes is that of dynamical scaling. Dynamical scaling is nicely illustrated through the behavior of the mean-square width of the surface or interface which for a substrate of linear size L scales as [22]

$$W^2(L,t) = L^{2\zeta} F(t/L^z), \tag{3}$$

where ζ is the roughness exponent and z is the dynamical exponent. For EW we have $z=2$ whereas for MH $z=4$. The value of ζ depends on whether correlated or uncorrelated noise is considered.

Langevin equations have the drawback that they do not really mirror the atomistic processes underlying the fluctuations of the interfaces and surfaces. In order to capture the physics on the microscopic level one commonly designs simple atomistic models (characterized by some specific deposition and/or diffusion rules) which are then often studied numerically. However, it is not always clear what the corresponding Langevin equation is. Usually, numerical simulations are used in order to extract the exponents ζ and z [see Eq. (3)] which generally permit to relate the microscopic model to one of the Langevin equations (universality classes). These exponents, however, encode only partly the information given by a scaling behavior, as the scaling functions, like $F(y)$ in Eq. (3), are themselves different for different universality classes.

In this paper we focus on two-point quantities as, for example, the space-time response and the space-time correlation functions which also display a dynamical scaling behavior. On a more fundamental level we show, using arguments first given in [23], that in systems described by Eqs. (1) and (2) the scaling functions of these two-point quantities can be derived by exclusively exploiting the symmetry properties of the underlying *noiseless*, i.e., deterministic, equations. This approach, which is based on generalized, space- and time-dependent, symmetries of the dynamical system [24,25], has in the past already been applied successfully in the special case $z=2$ to systems undergoing phase ordering [23,26–28] and to nonequilibrium phase transitions [29]. Here we show that local space-time symmetries also permit to fix (up to some numerical factors) the scaling functions of space-time response and correlation functions in cases where $z=4$. On a more practical level we demonstrate the usefulness of the scaling functions of these two-point quantities [which depend on two different space-time points $(\mathbf{x},t)$ and $(\mathbf{y},s)$] in the characterization of the universality classes of nonequilibrium growth processes. Whereas in the study of critical systems scaling functions, which are universal and characterize the different universality classes, are routinely investigated (and this both at equilibrium [30] and far from equilibrium [31,32]), in nonequilibrium growth processes the focus usually lies on simple quantities like, for example, the exponents ζ and z. There are some notable exceptions where scaling functions have been discussed (see, for example, [8,16]), but these studies were in general restricted to one-time quantities. However, also in nonequilibrium growth processes scaling functions of two-point functions are universal and should therefore be very valuable in the determination of the universality class of a given microscopic model or experimental system. We illustrate this by computing through Monte Carlo simulations the two-point space-time correlation function for two microscopic models which have been proposed to belong to the same universality class as the Edwards-Wilkinson equation (1) with Gaussian white noise [33–37].

The paper is organized in the following way. In Sec. II we discuss the EW and the MH equations in more detail and introduce the two-point functions. Section III is devoted to the computation of the exact expressions of the space-time correlation and response functions by Fourier transformation. These exact results show *inter alia* that the response of the system to the noise does not depend explicitly on the specific choice of the noise itself. In Sec. IV we discuss the space-time symmetries of the noiseless equations, whereas in Sec. V we show how these symmetries can be used for the derivation of the scaling functions of two-point functions. In Sec. VI we numerically study two microscopic growth models which have been proposed to belong to the Edwards-Wilkinson universality class. There has been a recent debate on the universality class of these models which we resolve by studying the scaling function of the space-time correlation function. Finally, in Sec. VII we give our conclusions. Some technical points are deferred to the Appendices.

II. NOISE MODELIZATION AND SPACE-TIME QUANTITIES

Our main interest in this paper is the investigation of space-time quantities in systems described by the quite general linear stochastic equations (1) and (2). Setting the mean growth velocity λ to zero (which can always be achieved by transforming into the comoving frame) both cases can be captured by the single equation

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\nu_{2l}(-\nabla^2)^l h(\mathbf{x},t) + \eta(\mathbf{x},t) \qquad (4)$$

with $l=1$ (EW) or $l=2$ (MH). As is well known, these equations of motion can be derived from a free field theory [7].

Depending on the physical context, different types of noise may be considered. For the EW case we shall discuss both Gaussian white noise (EW1)

$$\langle \eta(\mathbf{x},t) \rangle = 0, \quad \langle \eta(\mathbf{x},t)\eta(\mathbf{y},s) \rangle = 2D\delta^d(\mathbf{x}-\mathbf{y})\delta(t-s) \quad (5)$$

and spatially correlated noise (EW2)

$$\langle \eta(\mathbf{x},t) \rangle = 0, \quad \langle \eta(\mathbf{x},t)\eta(\mathbf{y},s) \rangle = 2D|\mathbf{x}-\mathbf{y}|^{2\rho-d}\delta(t-s) \quad (6)$$

with $0<\rho<d/2$. In the past, these two types of noise have been used in the modeling of nonequilibrium growth processes [4,5]. If, however, one wishes to model thermal equilibrium interface fluctuations, such as, for example, step fluctuations rate-limited by evaporation-condensation, one has to consider white noise with the Einstein relation $D=\nu_2 k_B T$ where T is the temperature and k_B the Boltzmann constant. For the MH case we also consider the noises (5) and (6), called MH1 and MH2 in the following. In this case, however, white noise can only be used in nonequilibrium situations as it breaks the conservation of mass encoded in the Langevin equation (2). We shall not consider here the noisy Mullins-Herring equation with conserved noise which assures the relaxation toward equilibrium of a system with conserved dynamics, as this is covered by a forthcoming publication [38].

Two-time quantities have been shown in many circumstances to yield useful insights into the dynamical behavior of systems far from equilibrium [32]. Of special interest are space- and time-dependent functions such as, for example, the space-time response $R(\mathbf{x},\mathbf{y},t,s)$ or the height-height space-time correlation

$$C(\mathbf{x},\mathbf{y},t,s) = \langle h(\mathbf{x},t)h(\mathbf{y},s) \rangle, \qquad (7)$$

where the brackets indicate an average over the realization of the noise. The space-time response, defined by

$$R(\mathbf{x},\mathbf{y},t,s) = \left.\frac{\delta\langle h(\mathbf{x},t)\rangle}{\delta j(\mathbf{y},s)}\right|_{j=0}, \qquad (8)$$

measures the response of the interface at time t and position $\mathbf{x}$ to a small perturbation $j(\mathbf{y},s)$ at an earlier time s and at a different position $\mathbf{y}$ [39]. For reasons of causality we have $t>s$. At the level of the Langevin equation the perturbation enters through the addition of j to the right-hand side. Assuming spatial translation invariance in the directions parallel to the interface, we have

$$C(\mathbf{x},\mathbf{y},t,s) = C(\mathbf{x}-\mathbf{y},t,s), \quad R(\mathbf{x},\mathbf{y},t,s) = R(\mathbf{x}-\mathbf{y},t,s). \qquad (9)$$

The autocorrelation and autoresponse functions are then defined by

$$C(t,s) := C(\mathbf{0},t,s), \quad R(t,s) := R(\mathbf{0},t,s). \qquad (10)$$

It is well known that the systems discussed here present a simple dynamical scaling bevahior [see Eq. (3)]. For example, for the autoresponse and the autocorrelation functions we have

$$R(t,s) \sim s^{-a-1} f_R(t/s), \quad C(t,s) \sim s^{-b} f_C(t/s), \qquad (11)$$

which defines the nonequilibrium exponents a and b. This terminology is well known from magnetic systems. Combining Eqs. (3), (7), and (11), one ends up with the scaling relation $b=-2\zeta/z$, which relates b to the known exponents ζ and z. In addition the scaling functions f_R and f_C define two additional exponents λ_R and λ_C by their asymptotic behavior

$$f_R(y) \overset{y\to\infty}{\sim} y^{-\lambda_R/z}, \quad f_C(y) \overset{y\to\infty}{\sim} y^{-\lambda_C/z}, \qquad (12)$$

where z is again the dynamical exponent introduced in the Introduction. Similarly, one obtains for the space-time quantities the following scaling forms:

$$R(\mathbf{x}-\mathbf{y},t,s) \sim s^{-a-1} F_R(|\mathbf{x}-\mathbf{y}|^z/s, t/s),$$

$$C(\mathbf{x}-\mathbf{y},t,s) \sim s^{-b} F_C(|\mathbf{x}-\mathbf{y}|^z/s, t/s). \qquad (13)$$

III. RESPONSE AND CORRELATION FUNCTIONS: EXACT RESULTS

This section is devoted to the computation of space-time quantities by directly solving the Langevin equations (1) and (2) in the physically relevant cases $d=1$ and $d=2$. These exact results will be used in the following in two different ways. In Sec. V we use these expressions in order to check whether our approach, which exploits exclusively the generalized space-time symmetries of the deterministic part of the equation of motion, yields the correct results. In addition, in Sec. VI we compare these expressions with the numerically determined scaling functions obtained for two different atomistic models in order to decide on the universality class of these models.

A. $z=2$: The Edwards-Wilkinson case

In order to compute the response of the surface/interface to a small perturbation we add the term $j(\mathbf{x},t)$ to the right-hand side of the Langevin equation and then go to reciprocal space. In the EW case the solution of the resulting equation is (with $d=1, 2$)

$$\hat{h}(\mathbf{k},t) = e^{-\nu_2 \mathbf{k}^2 t} \int_0^t dt' e^{\nu_2 \mathbf{k}^2 t'} [\hat{\eta}(\mathbf{k},t') + \hat{j}(\mathbf{k},t')], \qquad (14)$$

where we denote by $\hat{h}(\mathbf{k},t)$, $\hat{\eta}(\mathbf{k},t)$ resp. $\hat{j}(\mathbf{k},t)$ the Fourier transform of $h(\mathbf{x},t)$, $\eta(\mathbf{x},t)$ resp. $j(\mathbf{x},t)$. We prepare the system at time $t=0$ in an out-of-equilibrium state. For simplicity we assume flat initial conditions, i.e., $h(\mathbf{x},0)=0$, but our results are the same for any initial state with $\langle h(\mathbf{x},0)\rangle=0$ [40]. This preparation enables us to study the approach to equilibrium for the EW1 case with a valid Einstein relation. For the corresponding study of equilibrium dynamical properties (as encountered in the recent experiments on step fluctuations [10–12]) we have to prepare the system at $t=-\infty$ and replace in (14) the lower integration boundary 0 by $-\infty$. We shall in the following concentrate on the out-of-equilibrium situation.

Taking the functional derivative of (14) and transforming back to real space yields the result

$$R(\mathbf{x}-\mathbf{y},t,s) = \hat{r}_0 \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\nu_2 \mathbf{k}^2(t-s)} \qquad (15)$$

$$= r_0(t-s)^{-d/2} \exp\left(-\frac{(\mathbf{x}-\mathbf{y})^2}{4\nu_2(t-s)}\right) \qquad (16)$$

with $r_0 = \frac{1}{(2\sqrt{\pi\nu_2})^d}$ and $t>s$. The exponents a and λ_R as well as the scaling function f_R can readily be obtained from the expression of the autoresponse function [see Eqs. (11) and (12)]

$$R(t,s) = r_0(t-s)^{-d/2}, \qquad (17)$$

yielding $a=\frac{d}{2}-1$, $\lambda_R=d$, and $f_R(y)\sim(y-1)^{d/2-1}$.

The expression for the space-time response is completely independent from the choice of the noise term as long as $\langle\hat{\eta}(\mathbf{k},t)\rangle=0$. This also holds for the MH case, as discussed in the next subsection. In Sec. V we shall discuss an alternative way of looking at this fact.

A similar straightforward calculation yields for the space-time correlation the expression

$$C(\mathbf{x}-\mathbf{y},t,s) = \int_0^t dt' \int_0^s dt'' \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\nu_2 \mathbf{k}^2(t+s-t'-t'')}$$

$$\times \langle\hat{\eta}(\mathbf{k},t')\hat{\eta}(-\mathbf{k},t'')\rangle, \qquad (18)$$

where we have exploited the spatial translation invariance of the noise correlator. For *Gaussian white noise* we then obtain for the space-time correlation

$$C(\mathbf{x}-\mathbf{y},t,s) = c_0|\mathbf{x}-\mathbf{y}|^{2-d}\left[\Gamma\left(\frac{d}{2}-1,\frac{(\mathbf{x}-\mathbf{y})^2}{4\nu_2(t+s)}\right)\right.$$

$$\left. -\Gamma\left(\frac{d}{2}-1,\frac{(\mathbf{x}-\mathbf{y})^2}{4\nu_2(t-s)}\right)\right] \qquad (19)$$

with $c_0 = \frac{D}{2^{d+3}\pi^{3d/2}\nu_2}$. The autocorrelation function for $d\neq 2$ [41]

is obtained by using the known series expansion of the incomplete Gamma-functions Γ:

$$C(t,s) = \frac{2c_0(4\nu_2)^{1-d/2}}{2-d} s^{1-d/2} \left[\left(\frac{t}{s} + 1 \right)^{1-d/2} - \left(\frac{t}{s} - 1 \right)^{1-d/2} \right]$$

(20)

from which we find $b = \frac{d}{2} - 1$, $\lambda_C = d$, and $f_C(y) \sim (y+1)^{1-d/2} - (y-1)^{1-d/2}$. For the special case $d = 2$ one has to take the logarithmic behavior of the Gamma-functions into account which yields

$$C(t,s) = c_0 \ln \frac{t+s}{t-s}.$$

(21)

For *spatially correlated noise* the space-time correlator can only be written as a series expansion:

$$C(\mathbf{x} - \mathbf{y}, t, s) = \sum_{n=0}^{\infty} (-1)^n a_n^{(d)}(\rho) |\mathbf{x} - \mathbf{y}|^{2n} [(t+s)^{-(2n-2\rho+d-2)/2}$$
$$- (t-s)^{-(2n-2\rho+d-2)/2}]$$

(22)

with

$$a_n^{(d)}(\rho) = b_n^{(d)} \frac{2^{2\rho} D \Gamma(\rho) \Gamma(n - \rho + d/2)}{(2\rho - 2n - d + 2)\pi^{2-3d/2}\Gamma(d/2 - \rho)\nu_2^{n-\rho+d/2}}$$

and $b_n^{(1)} = \frac{1}{(2n)!}$, $b_n^{(2)} = \frac{1}{[(2n)!!]^2}$, whereas for the autocorrelator one gets

$$C(t,s) = a_0^{(d)}(\rho) s^{1-d/2+\rho} \left[\left(\frac{t}{s} + 1 \right)^{1-d/2+\rho} - \left(\frac{t}{s} - 1 \right)^{1-d/2+\rho} \right],$$

(23)

yielding $b = \frac{d}{2} - 1 - \rho$, $\lambda_C = d - 2\rho$, and $f_C(y) \sim (y+1)^{1-d/2+\rho} - (y-1)^{1-d/2+\rho}$.

Looking at the expressions (19) and (22), we see that in both cases the space-time correlation has the following scaling form:

$$C(\mathbf{x} - \mathbf{y}, t, s) = |\mathbf{x} - \mathbf{y}|^{\alpha} F\left(\frac{(\mathbf{x} - \mathbf{y})^2}{t+s}, \frac{(\mathbf{x} - \mathbf{y})^2}{t-s} \right)$$

(24)

with $\alpha = 2 - d$ resp. $2 - d + 2\rho$ for EW1 resp. EW2. The scaling function F is then a function of the two scaling variables $\frac{(\mathbf{x}-\mathbf{y})^2}{t+s}$ and $\frac{(\mathbf{x}-\mathbf{y})^2}{t-s}$. In the nonequilibrium situation we discuss here the space-time correlation function is therefore not time translation invariant. For equilibrium systems which are prepared at $t = -\infty$ time translation invariance is of course recovered. It is worth noting that the scaling form (24) corrects the scaling forms given in [3] where only a dependence on $\frac{(\mathbf{x}-\mathbf{y})^2}{t-s}$ was predicted far from equilibrium.

B. $z = 4$: The Mullins-Herring case

For the Mullins-Herring case we proceed along the same line as for the Edwards-Wilkinson case. We here only give the results for one-dimensional interfaces and refer the reader to the Appendix A for the two-dimensional case. The solu-

tion of the Langevin equation in Fourier space reads in the MH case

$$\hat{h}(\mathbf{k}, t) = e^{\nu_4 \mathbf{k}^4 t} \int_0^t dt' e^{-\nu \mathbf{k}^4 t'} [\hat{\eta}(\mathbf{k}, t') + \hat{j}(\mathbf{k}, t')]$$

(25)

which after some algebra yields for one-dimensional interfaces the expression

$$R(\mathbf{x} - \mathbf{y}, t, s) = \frac{1}{\pi \nu_4^{1/4}} [\nu_4(t-s)]^{-1/4} \left[\Gamma\left(\frac{5}{4} \right) \right.$$
$$\times {}_0F_2\left(\frac{1}{2}, \frac{3}{4}; \frac{(\mathbf{x}-\mathbf{y})^4}{256\nu_4(t-s)} \right) - 2\Gamma\left(\frac{3}{4} \right)$$
$$\times \left(\frac{(\mathbf{x}-\mathbf{y})^4}{256[\nu_4(t-s)]} \right)^{1/2} {}_0F_2\left(\frac{5}{4}, \frac{3}{2}; \frac{(\mathbf{x}-\mathbf{y})^4}{256\nu_4(t-s)} \right) \right]$$

(26)

for the space-time response. Here the ${}_0F_2$ functions are generalized hypergeometric functions. It is worth noting that exponentially growing contributions to the functions ${}_0F_2$ just cancel each other, yielding a response function which decreases for $(\mathbf{x}-\mathbf{y})^4/(t-s) \to \infty$, as it should.

The autoresponse function is straightforwardly found (with $t > s$):

$$R(t,s) = \frac{\Gamma\left(\frac{5}{4} \right)}{\pi \nu_4^{1/4}} (t-s)^{-1/4}$$

(27)

which gives us the quantities $a = -\frac{3}{4}$, $\lambda_R = 1$, and $f_R(y) \sim (y-1)^{-1/4}$. One straightforwardly verifies that in any space dimension d one has the relations $a = \frac{d}{4} - 1$, $\lambda_R = d$, and $f_R(y) \sim (y-1)^{-d/4}$.

As for the EW case we remark that the exact expressions (26) and (27) are independent of the noise: we obtain the same results for Gaussian white noise and for spatially correlated noise. Let us add that we have here only considered perturbations which are not mass conserving. Whereas this is physically sound for the cases we have in mind here, one usually considers mass conserving perturbations in the context of critical dynamics [7,38].

Turning to the correlation function, we proceed as for the EW case and obtain

$$C(\mathbf{x} - \mathbf{y}, t, s) = \int_0^t dt' \int_0^s dt'' \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\nu_4 \mathbf{k}^4(t+s-t'-t'')}$$
$$\times \langle \hat{\eta}(\mathbf{k}, t') \hat{\eta}(-\mathbf{k}, t'') \rangle.$$

(28)

For Gaussian white noise this then yields in $1+1$ dimensions the expressions

$$C(\mathbf{x} - \mathbf{y}, t, s) = \frac{D}{(2\pi)^2} \sum_{n=0}^{\infty} \frac{(-1)^n |\mathbf{x} - \mathbf{y}|^{2n}}{(2n)! \nu_4^{(2n+1)/4}(3-2n)} \Gamma\left(\frac{2n+1}{4} \right)$$
$$\times [(t+s)^{(3-2n)/4} - (t-s)^{(3-2n)/4}]$$

(29)

and

$$C(t,s) = \frac{D\Gamma(5/4)}{3\pi^2 \nu_4^{1/4}} [(t+s)^{3/4} - (t-s)^{3/4}] \tag{30}$$

for the space-time correlation and the autocorrelation functions. Similarly, for spatial correlated noise we have

$$C(\mathbf{x} - \mathbf{y}, t, s) = \sum_{n=0}^{\infty} (-1)^n \tilde{a}_n^{(1)}(\rho) |\mathbf{x} - \mathbf{y}|^{2n} [(t+s)^{-(2n-2\rho+d-4)/4}$$
$$- (t-s)^{-(2n-2\rho+d-4)/4}] \tag{31}$$

with

$$\tilde{a}_n^{(1)}(\rho) = \frac{2^{2\rho} D\Gamma(\rho)\Gamma((1+2n-2\rho)/4)}{(2n)!(2\rho - 2n + 3)\pi^{1/2}\Gamma((1-2\rho)/2)\nu_4^{(1+2n-2\rho)/4}},$$

the autocorrelation being given by the term with $n=0$.

IV. SPACE-TIME SYMMETRIES OF THE NOISELESS EQUATIONS

We have seen in the previous section that both models under consideration show dynamical (critical) scaling behavior. For critical systems it is well know [7,25] that the multipoint correlators $G(\mathbf{x}_1, t_1, \ldots, \mathbf{x}_n, t_n)$ $:= \langle h_1(\mathbf{x}_1, t_1), \ldots, h_n(\mathbf{x}_n, t_n) \rangle$ satisfy a covariance behavior of the kind

$$G(b\mathbf{x}_1, b^z t_1, \ldots, b\mathbf{x}_n, b^z t_n) = b^{x_1 + \ldots + x_n} G(\mathbf{x}_1, t_1, \ldots, \mathbf{x}_n, t_n)$$

with a constant rescaling factor b and with some numbers x_i. It has been proposed by Henkel [25] to extend this dynamical scaling with b constant to rescaling factors which are space and time dependent, i.e., $b \to b(\mathbf{x}, t)$. For the special case $z=2$ this can be connected to so-called Schrödinger invariance [42,43]. In [24,25] the concept of Schrödinger invariance has been extended to the case of an arbitrary value of z, but a concrete comparison of the space-time quantities predicted by this theory with specific, exactly solvable models with $z \neq 2$ is still lacking. Here we wish to apply this theory—which is called theory of local scale invariance (LSI)—not only to the case $z=2$ (i.e., the EW case), but also to the MH case with $z=4$.

LSI proposes to derive expressions for the scaling functions of two-time quantities by looking at the space-time symmetries of the *noiseless* equations of motion. In our case the deterministic equations for $z=2$ resp. $z=4$ are obtained by dropping the noise term on the right-hand side of Eq. (1) resp. (2). The case $z=2$ then yields the free diffusion equation or, equivalently, when going to complex times, the free Schrödinger equation. The maximal kinematic group of space-time symmetries which leave the free Schrödinger equation invariant is the so-called Schrödinger group [42–45], the elements of which transform space and time in the following way:

$$\mathbf{x} \to \mathbf{x}' = \frac{\mathcal{R}\mathbf{x} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta}, \quad t \to t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha\delta - \beta\gamma = 1, \tag{32}$$

where $\mathcal{R}$ is a rotation matrix, whereas $\mathbf{v}$, $\mathbf{a}$ and α, β, γ, δ are real parameters. We write for this also $(\mathbf{x}', t') = g(\mathbf{x}, t)$ and

denote the inverse transformation by $g^{-1}(\mathbf{x}', t')$. Under the action of these group elements solutions Ψ of the Schrödinger equation transform as

$$\Psi \to \Psi' = f_g(g^{-1}(\mathbf{x}, t))\Psi(g^{-1}(\mathbf{x}, t)), \tag{33}$$

where the companion function f_g is known explicitly [43]. The generators of the Schrödinger group, which can be considered as infinitesimal version of these transformations, form a Lie-Algebra.

In [25] generators of space-time transformations have been constructed which act as dynamical symmetries on a more general deterministic equation, namely on

$$\left[-\lambda\partial_t + \frac{1}{z^2}\partial_r^z \right]\Psi = 0. \tag{34}$$

where $z > 0$ is a real number. It is easy to see that Eq. (34) is equivalent to (1) for $z=2$ resp. to (2) for $z=4$ when setting $\nu_2 = (4\lambda)^{-1}$ resp. $\nu_4 = -(16\lambda)^{-1}$. The generators of the corresponding Lie algebra are explicitly given by (for $d=1$) [25]

$$X_{-1} = -\partial_t, \tag{35}$$

$$Y_{-\theta} = -\partial_r, \tag{36}$$

$$X_0 = -t\partial_t - \frac{1}{z}r\partial_r - \frac{x}{z}, \tag{37}$$

$$X_1 = -t^2\partial_t - \frac{2}{z}tr\partial_r - \frac{2x}{z}t - \lambda r^2\partial_r^{2-z} - 2\gamma_1(2-z)r\partial_r^{1-z}$$
$$- \gamma_1(2-z)(1-z)\partial_r^{-z}, \tag{38}$$

$$Y_{-1/z+1} = -t\partial_r - \lambda zr\partial_r^{2-z} - \gamma_1 z(2-z)\partial_r^{1-z}. \tag{39}$$

Operators like ∂_r^{1-z} are so-called fractional derivatives, which we recall in Appendix B. Hereby the quantities x and γ_1 are related by

$$x = \frac{z-1}{2} + \frac{\gamma_1}{\lambda}(2-z). \tag{40}$$

As shown in [25] these space-time symmetries (35)–(39) can be used to fix the form of the two-time response function completely. Using all generators and writing $\mathbf{r} = \mathbf{x} - \mathbf{y}$, one obtains

$$R_0(\mathbf{r}, t, s) = (t-s)^{-2x/z}\phi\left(\frac{|\mathbf{r}|}{(t-s)^{1/z}}\right), \tag{41}$$

where the index 0 indicates that this is the result for the noise-free theory. The scaling function $\phi(u)$ satisfies the fractional differential equation [25]

$$[\partial_u + z\lambda u\partial_u^{2-z} + 2z(2-z)\gamma_1\partial_u^{1-z}]\phi(u) = 0. \tag{42}$$

We have to stress that the scaling function given in [25] is not the most general solution of this equation. In Appendix C we derive this most general solution for any rational z. As shown in the next section it is this solution which permits us to derive the exact expressions for the space-time response and correlation functions in the MH case by exploiting ex-

clusively the space-time symmetries of the deterministic equation (2). For $z=2$ we recover the known result [25]

$$\phi(u) = \phi_0 \exp(-\lambda u^2), \qquad (43)$$

where the numerical factor ϕ_0 is not fixed by the theory. For the case $z=4$ and $d=1$ our new result is (see Appendix C for the expression obtained in two space dimensions)

$$\phi(u) = \widetilde{c}_0 \left(-\frac{\lambda}{16}u^4\right)^{1/4} {}_0F_2\left(\frac{3}{4},\frac{5}{4},-\frac{\lambda}{16}u^4\right) + \widetilde{c}_1\left(-\frac{\lambda}{16}u^4\right)^{1/2}$$
$$\times {}_0F_2\left(\frac{5}{4},\frac{3}{2},-\frac{\lambda}{16}u^4\right) + \widetilde{c}_2 \, {}_0F_2\left(\frac{1}{2},\frac{3}{4},-\frac{\lambda}{16}u^4\right) \quad (44)$$

with some constants $\widetilde{c}_0$, $\widetilde{c}_1$, and $\widetilde{c}_2$. Here we have used the fact that $x=\frac{d}{2}$ in a free field theory as is easily obtained from a dimensional analysis (see also the next section). The coefficients $\widetilde{c}_0$, $\widetilde{c}_1$, and $\widetilde{c}_2$ have to be arranged in such a way that $\phi(u)$ vanishes for $u \to \infty$. Analyzing the leading terms [46,47] one realizes that the condition

$$\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)\widetilde{c}_0 + \Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{2}\right)\widetilde{c}_1 + \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\widetilde{c}_2 = 0 \tag{45}$$

provides exactly this, as it cancels all exponentially growing terms.

Let us close this section by noting that in the derivation of expression (41) we exploited the fact that the exact responses of both the EW and MH case are time translation invariant. Often when discussing out-of-equilibrium systems this is not the case and one has to consider the subalgebra where the generator X_{-1}, responsible for time translation invariance, is omitted [23–25,27,28]. This then yields for the autoresponse the expression

$$R_0(t,s) = r_0 s^{-1-a}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/z}\left(\frac{t}{s}-1\right)^{-1-a'},$$

where the parameters a and a' have to be determined by comparing with known results. When setting $a=a'=\lambda_R/z-1$, one recovers our result.

V. DETERMINATION OF RESPONSE AND CORRELATION FUNCTIONS FROM SPACE-TIME SYMMETRIES

In order to use the symmetry considerations of the last section, we have to adopt the standard field theoretical setup for the description of Langevin equations [23,49,50]. Apart from the field $h(\mathbf{x},t)$ we consider the so-called response field $\widetilde{h}(\mathbf{x},t)$ which leads to the action

$$S[h,\widetilde{h}] = \int du \, d\mathbf{R}[\widetilde{h}(\partial_u + \nu_{2l}(-\nabla^2)^l)h]$$
$$+ \frac{1}{2}\int du \, d\mathbf{R} \, du' \, d\mathbf{R}'\widetilde{h}(\mathbf{R},u)$$
$$\times \langle \eta(\mathbf{R},u)\eta(\mathbf{R}',u')\rangle\widetilde{h}(\mathbf{R}',u'), \quad (46)$$

where $l=1$ for the EW case and $l=2$ for the MH case. The

temporal integration is from 0 to ∞ whereas the spatial integration is over the whole space. The following reduction to the noise-free theory is a well-known procedure [23] but we recall the most important steps in order to establish notations.

Varying the action yields the equation of motion for the fields h and $\widetilde{h}$

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\nu_{2l}(-\nabla^2)^l h(\mathbf{x},t) - \int du d\mathbf{R}\widetilde{h}(\mathbf{R},u)$$
$$\times \langle \eta(\mathbf{R},u)\eta(\mathbf{x},t)\rangle, \tag{47}$$

$$\frac{\partial \widetilde{h}(\mathbf{x},t)}{\partial t} = \nu_{2l}(-\nabla^2)^l\widetilde{h}(\mathbf{x},t). \tag{48}$$

As expected, one recovers for the height $h(\mathbf{x},t)$ Eqs. (1) and (2) by identifying the noise with the term $-\int du \, d\mathbf{R}\widetilde{h}(\mathbf{R},u) \times \langle \eta(\mathbf{R},u)\eta(\mathbf{x},t)\rangle$.

One can now proceed by looking at the multipoint functions which are defined in the usual way via functional integrals:

$$\left\langle \prod_{i=1}^{n} h(\mathbf{x}_i,t_i) \prod_{j=n+1}^{m} \widetilde{h}(\mathbf{x}_j,t_j) \right\rangle$$
$$:= \int \mathcal{D}[h]\mathcal{D}[\widetilde{h}]\prod_{i=1}^{n} h(\mathbf{x}_i,t_i) \prod_{j=n+1}^{m} \widetilde{h}(\mathbf{x}_j,t_j)\exp(-S[h,\widetilde{h}]). \tag{49}$$

Within this formalism the space-time response (8) is given by

$$R(\mathbf{x},\mathbf{y},t,s) = \langle h(\mathbf{x},t)\widetilde{h}(\mathbf{y},s)\rangle. \tag{50}$$

In order to proceed one splits up the action in the same way as done in [23], that is as

$$S[h,\widetilde{h}] = S_0[h,\widetilde{h}] + S_{th}[h,\widetilde{h}] \tag{51}$$

with the deterministic part

$$S_0[h,\widetilde{h}] = \int du \, d\mathbf{R}[\widetilde{h}(\mathbf{R},u)(\partial_u + \nu_{2l}(-\nabla^2)^l)h(\mathbf{R},u)] \tag{52}$$

and the noise part

$$S_{th}[h,\widetilde{h}] = \frac{1}{2}\int du \, d\mathbf{R} \, du' \, d\mathbf{R}'\widetilde{h}(\mathbf{R},u)$$
$$\times \langle \eta(\mathbf{R},u)\eta(\mathbf{R}',u')\rangle\widetilde{h}(\mathbf{R}',u'). \tag{53}$$

We call the theory exclusively described by S_0 noise-free and denote averages with respect to this theory with $\langle\cdots\rangle_0$. The n-point functions of the full theory can then be written as

$$\left\langle \prod_{i=1}^{n} h(\mathbf{x}_i, t_i) \prod_{j=n+1}^{m} \widetilde{h}(\mathbf{x}_j, t_j) \right\rangle = \left\langle \prod_{i=1}^{n} h(\mathbf{x}_i, t_i) \prod_{j=n+1}^{m} \widetilde{h}(\mathbf{x}_j, t_j) \exp(-S_{th}[h, \widetilde{h}]) \right\rangle_0 . \qquad (54)$$

It is easy to see that the noise-free theory has a Gaussian structure both for the EW and the MH model. Introducing the two-component field $\Psi = \binom{h}{\widetilde{h}}$ one can write the exponential $\exp(-S_0[h, \widetilde{h}])$ as $\exp(-\int du \, d\mathbf{r} \, du' \, d\mathbf{r}' \, \Psi^t \mathbf{A} \Psi)$ with

$$\mathbf{A} = \frac{1}{2} \begin{pmatrix} 0 & \delta(u-u')\delta(\mathbf{r}-\mathbf{r}')[(-\nabla^2)^l - \partial_u] \\ \delta(u-u')\delta(\mathbf{r}-\mathbf{r}')[(-\nabla^2)^l + \partial_u] & 0 \end{pmatrix}. \qquad (55)$$

From this one deduces two important facts which we will need in the sequel. First, one has

$$\langle \underbrace{h \ldots h}_{n} \underbrace{\widetilde{h} \ldots \widetilde{h}}_{m} \rangle_0 = 0 \qquad (56)$$

unless $n=m$, which is due to the antidiagonal structure of $\mathbf{A}$ (see for instance [7], Chap. 4) For $z=2$, Eq. (56) coincides with the Bargmann superselection rule [51]. Second, Wick's theorem holds. With this it follows that one can write the four-point function as

$$\langle h(\mathbf{x},t) h(\mathbf{y},s) \widetilde{h}(\mathbf{R},u) \widetilde{h}(\mathbf{R},u') \rangle_0$$
$$= \langle h(\mathbf{x},t) \widetilde{h}(\mathbf{R},u) \rangle_0 \langle h(\mathbf{y},s) \widetilde{h}(\mathbf{R}',u') \rangle_0$$
$$+ \langle h(\mathbf{y},s) \widetilde{h}(\mathbf{R},u) \rangle_0 \langle h(\mathbf{x},t) \widetilde{h}(\mathbf{R}',u') \rangle_0, \qquad (57)$$

where we have used Eq. (56).

Now one can calculate the quantities of interest, namely the space-time response and correlation functions. For this one develops the exponential in (54) in a power series. One remarks immediately that due to the selection rule (56) one has

$$R(\mathbf{x},\mathbf{y},t,s) = R_0(\mathbf{x},\mathbf{y},t,s) := \langle h(\mathbf{x},t) \widetilde{h}(\mathbf{y},s) \rangle_0, \qquad (58)$$

i.e., the linear response function of the full theory is *equal* to the noiseless linear response function. It follows that for any realization of the noise one gets the same expression for the response function, in agreement with the exact results derived in Sec. III. It is also worth noting that within a free field theory nonlinear responses vanish due to the same superselection rule. Things are of course more tricky for a field theory which is not free as here the noise can contribute to the response. Whether this is the case depends on the concrete form of the interaction.

By expanding the exponential in (54) we obtain in a similar way that the space-time correlation function is given by the expression

$$C(\mathbf{x},\mathbf{y},t,s) = \int du \, d\mathbf{R} \, du' \, d\mathbf{R}' \langle h(\mathbf{x},t) h(\mathbf{y},s) \widetilde{h}(\mathbf{R},u)$$
$$\times \widetilde{h}(\mathbf{R}',u') \rangle_0 \langle \eta(\mathbf{R},u) \eta(\mathbf{R}',u') \rangle. \qquad (59)$$

Using Wick's theorem we can replace the four-point function by two-point functions [see Eq. (57)] and obtain

$$C(\mathbf{x},\mathbf{y},t,s) = 2 \int du \, d\mathbf{R} \, du' \, d\mathbf{R}' \langle \eta(\mathbf{R},u) \eta(\mathbf{R}',u') \rangle$$
$$\times \langle h(\mathbf{x},t) \widetilde{h}(\mathbf{R},u) \rangle_0 \langle h(\mathbf{y},s) \widetilde{h}(\mathbf{R}',u') \rangle_0. \qquad (60)$$

Inspection of Eqs. (58) and (60) reveals that the only remaining undetermined quantity is the two-point function $\langle h(\mathbf{x},t) \widetilde{h}(\mathbf{y},s) \rangle_0$. However, as discussed in the previous section, this two-point function is fully determined by the space-time symmetries of the deterministic equation of motion. It remains to show that the insertion of this two-point function into Eqs. (58) and (60) indeed yields the exact expressions for the space-time quantities both for the EW and MH cases.

A. $z=2$: The Edwards-Wilkinson case

This case with Gaussian white noise has already been discussed in [23] in the context of phase ordering kinetics and of critical dynamics. The response function can be read off from Eq. (41) after inserting the scaling function (43). Recalling that for the EW case we have $z=2$, $x=\frac{d}{2}$, and $\nu_2 = (4\lambda)^{-1}$ we readily obtain the exact result (16). The only quantity left free by the theory is the numerical prefactor ϕ_0.

The space-time correlation function is obtained by inserting the same two-point function into Eq. (60). This is most easily seen by using the integral representation (15) of the two-point function which yields after interchanging the order of integration

$$C(\mathbf{x},\mathbf{y},t,s) = c_0 \int_0^t dt' \int_0^s dt'' \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\nu_2 \mathbf{k}^2(t-t')}$$
$$\times e^{-\nu_2 \mathbf{k}^2(s-t'')} \langle \hat{\eta}(\mathbf{k},t') \hat{\eta}(-\mathbf{k},t'') \rangle \qquad (61)$$

which is exactly the same expression (up to the undetermined constant c_0) as (18), and this for any choice of the noise correlator $\langle \hat{\eta}(\mathbf{k},t') \hat{\eta}(-\mathbf{k},t'') \rangle$. It then immediately follows that we recover the exact results for C given in Sec. III A.

B. $z=4$: The Mullins-Herring case

For the MH case we proceed along the same lines as for the EW case. Let us start with the one-dimensional case $d=1$. As already seen for the EW model, the response function is just the response function of the noise-free theory whose scaling function is given by (44) together with the condition

(45) needed for a response which vanishes for $u \rightarrow \infty$. In order to proceed further we remark that in a powers series expansion of (44) odd powers of the scaling variable u only enter through the term with coefficient $\widetilde{c}_0$. However, odd powers of the scaling variable are absent in the exact result (26), so we have to set $\widetilde{c}_0 = 0$. From Eq. (45) it then follows that

$$\Gamma\left(\frac{5}{4}\right)\widetilde{c}_1 = -2\Gamma\left(\frac{3}{4}\right)\widetilde{c}_2. \qquad (62)$$

Recalling that for the MH case $z=4$, $x=\frac{d}{2}$, and $\nu_4 = -16(\lambda)^{-1}$, it is now easy to check that the proposed scaling function together with (62) indeed yields the exact result (26) up to the normalization constant $\widetilde{c}_2$. This so determined two-point function can then be inserted into the correlation function (60), yielding the exact result (28). This is again most easily seen by using the integral representation

$$R(\mathbf{x}-\mathbf{y},t,s) = \hat{r}_0 \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\nu_4 \mathbf{k}^4(t-s)} \qquad (63)$$

of the response function (26).

In two dimensions we have to replace the expression (44) by (C16). It follows that in $d=2$ the scaling function obtained from LSI only contains two parameters which are furthermore related through the condition (C17). We are therefore left with a single undetermined parameter which only appears as a numerical prefactor, similar to the EW case. Inserting the resulting scaling function into the expressions (58) and (60) readily yields the exact results for the space-time quantities in two dimensions.

VI. MICROSCOPIC GROWTH MODELS AND SPACE-TIME CORRELATIONS

Many theoretical studies of growth processes focus on atomistic models where particles are deposited on a surface and are then incorporated into the growing surface following some specific rules which might include local diffusion processes. Of special interest is the determination of the universality class to which these models belong. This is usually achieved by computing some universal quantities through numerical simulations and comparing them to the corresponding quantities obtained from continuum growth equations like the EW and the MH equations discussed in this paper. In a commonly used approach one focuses on the estimation of the exponents z and ζ, which govern the behavior of the surface width (3), through the best data collapse.

In order to show that it is useful to look at two-time quantities in nonequilibrium growth processes we discuss in the following the space-time correlation function in the Family model [33] and in a variant of this model [36]. Even so these are very simple models, there is still some debate on the universality class to which these models belong, especially in $2+1$ dimensions. Whereas earlier numerical studies yielded the value $z=2$ for the dynamical exponent in the $2+1$-dimensional Family model [33–35,52], in agreement with the EW universality class with Gaussian white noise,
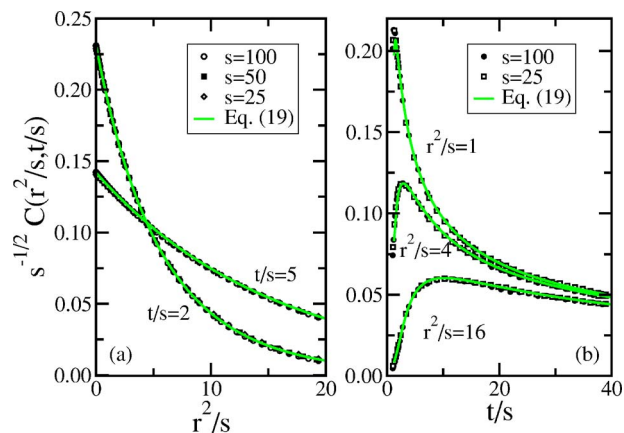

FIG. 1. (Color online) Dynamical scaling of the space-time correlation function $C(r^2/s, t/s)$ for the Family model in $1+1$ dimensions with different values of the waiting time s: (a) C vs r^2/s for some fixed values of t/s, (b) C vs t/s for some fixed values of r^2/s. The green curves are obtained from the exact result (19) derived from the continuum EW equations with uncorrelated Gaussian white noise. Numerical error bars are smaller than the sizes of the symbols.

Pal *et al.* [36,37] in their study obtained a value $z \approx 1.65$, pointing to a different universality class. In addition they studied a variant of this model (which we call restricted Family model in the following) for which they recovered $z = 2$. These results of Pal *et al.* are surprising, especially so as Vvedensky succeeded [53] in deriving in $1+1$ dimensions the EW equation with Gaussian white noise from both the Family and the restricted Family model through a coarse-graining procedure.

The Family model is a ballistic deposition model with surface diffusion where a particle is dropped at a randomly chosen surface site. Instead of fixing itself at this site, the particle first explores the local environment (usually one restricts this exploration to the nearest neighbors) and fixes itself at the lattice site with the lowest height. When two or more lattice sites other than the originally selected site have the same lowest height, one of these sites is selected randomly. In case the originally chosen lattice site is among the sites with the lowest height, the particle remains at this site. In the restricted version of this model, introduced in [36], the particle only moves to a site of lowest height when it is unique. This change has the effect that the moving of the particle only contributes deterministically to the surface shape.

We have simulated these two models both in $1+1$ and $2 +1$ dimensions. For the $1+1$-dimensional models all previous studies agree that $z=2$ and that both models belong to the one-dimensional EW universality class with Gaussian white noise. Our main interest here is the height-height space-time correlation function $C(\mathbf{x}-\mathbf{y},t,s)$. From the exact results presented in the first part of the paper we conclude that this two-time quantity should only depend on the two scaling variables t/s and r^2/s where $r=|\mathbf{x}-\mathbf{y}|$. In Fig. 1 we test this expected scaling behavior in the $1+1$-dimensional Family model. In Fig. 1(a) we fix t/s and plot the correlation function as a function of r^2/s, whereas in Fig. 1(b) r^2/s is

TABLE I. Estimates for the nonuniversal constants D and ν_2.

	D	ν_2
$d=1$	8.85(4)	1.260(6)
$d=1$ restricted	9.25(5)	1.312(7)
$d=2$	83(2)	1.49(5)
$d=2$ restricted	271(5)	2.63(6)

fixed and C is plotted vs t/s. Lattices with 12 800 sites have been simulated and the data shown result from averaging over 1000 runs with different random numbers. The curves obtained for different values of the waiting time s collapse on a common master curve when multiplying C with $s^{-1/2}$. In addition, these master curves nicely agree with the expression (19) obtained from the EW equation with uncorrelated white noise, once the nonuniversal constants D and ν_2 have been determined [56]. A similar good agreement is obtained for the restricted Family model. We list our estimates for D and ν_2 for both one-dimensional models in Table I. It follows from this table that D and ν_2 slightly differ in both models. In addition, D/ν_2 is slightly larger for the restricted model, even so the error bars are overlapping.

It is worthwhile noting that the continuum description is not expected to completely describe the lattice models for small values of r^2/s, as here the discrete nature of the lattice cannot be neglected any more. Deviations are indeed observed in Fig. 1(b) for $r^2/s=1$ and small values of t/s. These lattice effects are expected to get stronger when the dimensionality of the system increases.

After having verified that the computed scaling functions in both versions of the $1+1$-dimensional Family model agree with the solution of the EW continuum equation, let us now proceed to the more controversial $2+1$-dimensional case. In Fig. 2 we display the space-time correlation computed for the original Family model in $2+1$ dimensions. Again, in the left panel we fix t/s, whereas in the right panel r^2/s is kept constant. The data shown here have been obtained for lattices with 300×300 sites with 5000 runs for every waiting time. We carefully checked that our nonequilibrium data are not
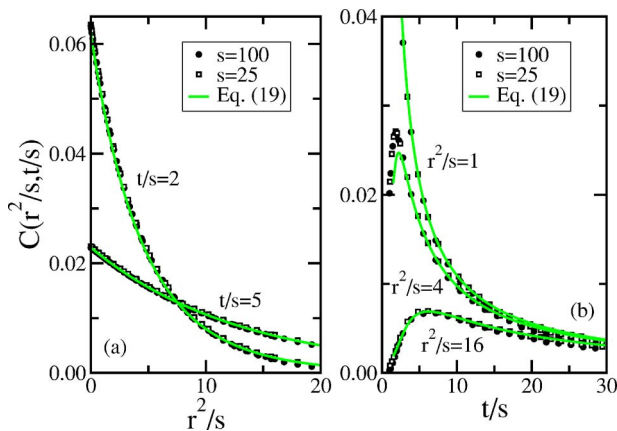

FIG. 2. (Color online) The same as in Fig. 1, but now for the Family model in $2+1$ dimensions. Numerical error bars are comparable to the sizes of the symbols.
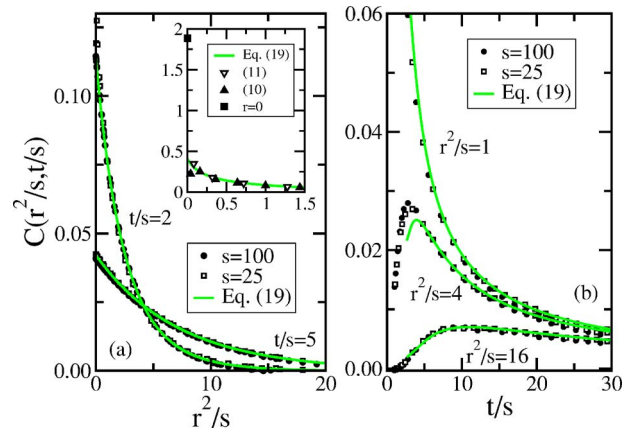

FIG. 3. (Color online) The same as in Fig. 2, but now for the restricted Family model in $2+1$ dimensions. The inset in (a) shows the correlation function in the (10) and (11) directions for the case $s=25$ and $t/s=1.04$. The change of the diffusion rule has a strong impact on the autocorrelation with $r=0$ and on the nearest-neighbor correlations. Numerical error bars are comparable to the sizes of symbols.

affected by finite-size effects. Furthermore, we ran different simulations with different random number generators and obtained the same results within error bars. We obtain as the main result of these simulations that the scaling function of the space-time correlation function is in excellent agreement (once the values of the nonuniversal constants have been determined, see Table I) with the exact result obtained from solving the two-dimensional EW equation with uncorrelated white noise. The only discrepancies observed for small r^2/s and small t/s are qualitativly the same as for the $1+1$ case and are explained by the discrete nature of the lattice. Our results are in accordance with the results of [33–35,52,53] but strongly disagree with those of Pal *et al.* [36,37]. Indeed, a noninteger value of z in a continuum description cannot be realized in a linear stochastic differential equation and leads to completely different scaling functions as those obtained from the EW equation.

In Fig. 3 (see also Table I) we show our results for the restricted family model in $2+1$ dimensions. Again, dynamical scaling is observed, and again the data are well described by the EW scaling functions in the scaling limit. However, the determined values of the nonuniversal quantities D and ν_2 are markedly different from the values obtained for the original model. Specifically, the ratio D/ν_2 (which is of the dimension k_BT) is much larger for the restricted model. Identifying D/ν_2 with a (nonequilibrium) temperature, we can view the processes in the restricted model to take place at a higher temperature than in the original model. This is in agreement with the observation from Pal *et al.* [37] that the surface is locally rougher in the restricted model, as evidenced by the larger value of the interface width. In addition, the change in the diffusion rule leads to a nonmonotonous behavior of the correlation function for small r^2/s, as shown in the inset of Fig. 3(a) for $s=25$ and $t/s=1.04$. Plotting the correlation function in both the (10) and (11) directions, we see that correlations between nearest neighbors are suppressed, whereas the autocorrelation, i.e., the correlation with

$r=0$, is strongly enhanced. This behavior can be understood by recalling that in the restricted model a particle only diffuses to a lower nearest-neighbor site when this site is unique, but otherwise remains on the original site. If we increase s and r, this effect weakens, and it completely vanishes in the scaling limit of large waiting times and large values of r^2/s.

From our observation that the numerically computed space-time correlation functions of both microscropic models coincide in the scaling limit with the exact expression from the EW model we conclude that both the Family model and the restricted Family model belong to the EW universality class with uncorrelated noise, and this not only in $1+1$ dimensions but also in $2+1$ dimensions.

VII. CONCLUSIONS

The aim of the present paper is twofold: on the one hand we discuss the usefulness of universal scaling functions of space-time quantities in characterizing the universality class of nonequilibrium growth models, on the other hand we demonstrate how a general symmetry principle allows to derive scaling functions of two-point quantities for equilibrium and nonequilibrium processes described by linear stochastic Langevin equations.

In the context of nonequilibrium growth processes it is rather uncommon to study universal scaling functions of two-point quantities in the dynamical scaling limit in order to determine the universality class to which a given microscopic model belongs. We have illustrated the usefulness of this approach by comparing the numerically obtained space-time correlation functions for two atomistic growth models with the exact expressions obtained from the corresponding continuum stochastic Langevin equation. This approach has allowed us to show that both models belong to the same universality class, thus correcting conclusions obtained in earlier numerical studies.

The study of universal scaling functions of space-time quantities should also be of value in more complex growth processes which are no more described by linear stochastic differential equations. Examples include ballistic deposition with an oblique incident particle beam [13] or growth processes of the Kardar-Parisi-Zhang (KPZ) [54] and related [55] universality classes where nonlinear effects can no more be neglected. In addition, the scaling functions studied here can also be measured in experiments involving nonequilibrium or equilibrium interface fluctuations. A promising system is given by equilibrium step fluctuations [10–12,19], as these are again described by linear Langevin equations.

In addition we have shown that in nonequilibrium growth processes scaling functions of out-of-equilibrium quantities can be derived in a *model independent* way by exploiting the generalized space-time symmetries of the *noiseless* part of the stochastic equations of motion. We have demonstrated explicitly how to proceed in the case of a rational dynamical exponent z, following the general ideas formulated by Henkel a few years ago [25]. In the context of nonequilibrium growth processes, the cases $z=2$ (EW model) and $z=4$ (MH model) are of special interest. The case $z=2$ has already been

studied extensively in the past. For the case $z=4$, however, we present, to our knowledge, for the first time, the derivation of nonequilibrium scaling functions by exploiting the mentioned symmetry principles. As these scaling functions are found to agree with the exact expressions derived from the MH equation, we conclude that the postulated space-time symmetries and the proposed way for constructing the scaling functions can also be valid for other cases than merely the case $z=2$.

Let us end this paper by a general remark on the applicability of the concept of local scale transformations in the context of other out-of-equilibrium processes. The data presented in this paper for the microscopic growth models nicely show the limitations of the continuum equations in describing atomistic models. Whereas in the limit of large times and large spatial separations the numerically computed correlation functions completely agree with the exact results from the continuum equation, notable deviations are observed for small times and small spatial separations. These deviations reflect the microscopic details of the models (underlying lattice structure, diffusion rules, etc.) which are not captured by the continuum model. This sheds an interesting light on an ongoing discussion [48,57–59] on the applicability of the theory of local scale invariance, which, we recall, permits to derive expressions for scaling functions starting from the noiseless part of the *continuum* equations of motion. Clearly, it has to be expected that the so-derived scaling functions cannot fully describe microscopic models in the short time and short distance limit. As it is exactly this limit which is in the center of the mentioned discussion involving the theory of local scale invariance, it seems advisable to take any observed deviations in this limit between numerical data, obtained from simulations of microscopic models, and the theoretically derived scaling functions *cum grano salis*, as these deviations might only reflect the microscopic nature of the models.

APPENDIX A: EXACT RESULTS FOR THE MULLINS-HERRING CASE IN $d=2$

We here compile the exact results in two space dimensions for the dynamical two-time quantities for the Mullins-Herring case with $z=4$. The space-time response can again be expressed through generalized hypergeometric functions and reads

$$
R(\mathbf{x}-\mathbf{y},t,s) = \frac{1}{8\pi\nu_4^{1/2}}\left[\pi^{1/2}\,_0F_2\left(\frac{1}{2},1,\frac{1}{256\nu_4}\left(\frac{|\mathbf{x}-\mathbf{y}|}{(t-s)^{1/4}}\right)^4\right)\right.
$$
$$
-\frac{1}{4\nu_4^{1/2}}\left(\frac{|\mathbf{x}-\mathbf{y}|}{(t-s)^{1/4}}\right)^2
$$
$$
\left.\times\,_0F_2\left(1,\frac{3}{2},\frac{1}{256\nu_4}\left(\frac{|\mathbf{x}-\mathbf{y}|}{(t-s)^{1/4}}\right)^4\right)\right]. \tag{A1}
$$

For the autoresponse one gets

$$R(t,s) = \frac{1}{8\pi^{1/2}\nu_4^{1/2}}(t-s)^{-1/2}. \tag{A2}$$

As already noted before, these expressions give the response of the system to the noise itself and therefore do not depend on the concrete realization of the noise as long as it is non-conserving.

For the space-time correlation function $C(\mathbf{x}-\mathbf{y},t,s)$ and for the autocorrelation function $C(t,s)$ we obtain the following expressions, which depend on the form of the noise.

Gaussian white noise (MH1):

$$C(\mathbf{x}-\mathbf{y},t,s) = \frac{D}{32\pi^2}\left[\left(\sum_{n=0,n\neq 1}^{\infty}\frac{(-1)^n|\mathbf{x}-\mathbf{y}|^{2n}\Gamma\left(\frac{n+1}{2}\right)}{(2n)!\Gamma(1+2n)\left[\Gamma\left(\frac{1}{2}-n\right)\right]^2\nu_4^{(n+1)/2}(1-n)}\cdot\left[(t+s)^{(1-n)/2}-(t-s)^{(1-n)/2}\right]\right)-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\pi\nu_4}\ln\left(\frac{t+s}{t-s}\right)\right], \tag{A3}$$

$$C(t,s) = \frac{D}{32\pi^{5/2}\nu_4^{1/2}}[(t+s)^{1/2}-(t-s)^{1/2}]. \tag{A4}$$

Spatially correlated noise (MH2):

$$C(\mathbf{x}-\mathbf{y},t,s) = \sum_{n=0}^{\infty}(-1)^n\tilde{a}_n^{(2)}(\rho)|\mathbf{x}-\mathbf{y}|^{2n}[(t+s)^{-(2n-2\rho-2)/4}$$
$$-(t-s)^{-(2n-2\rho-2)/4}], \tag{A5}$$

$$C(t,s) = \tilde{a}_0^{(2)}(\rho)[(t+s)^{(2+2\rho)/4}-(t-s)^{(2+2\rho)/4}] \tag{A6}$$

with

$$\tilde{a}_n^{(2)}(\rho) = \frac{2^{2\rho-1}D\Gamma(\rho)\Gamma((2+2n-2\rho)/4)}{2^{2n}(n!)^2(2\rho-2n+2)\Gamma(1-\rho)\nu_4^{(2+2n-2\rho)/4}}.$$

APPENDIX B: ON FRACTIONAL DERIVATIVES

We list here the most important properties of the fractional derivates as these will be used in Appendix C in the derivation of the scaling function of the space-time response. We stress the point that there are several definitions of fractional derivatives available, which are not equivalent. However, we need a special type of fractional derivative. We simply quote the most important properties as given in [25] and refer the reader to this reference for a more thorough introduction.

∂_r^a acts on a function $f(r)$ which can be expanded into the form $f(r)=\sum_{e\in E}f_e r^e+\sum_{n=0}^{\infty}F_n\delta^{(n)}(r)$. Here E is the set $E=\mu\mathbb{N}+\lambda$ with $\mu>0$ and $\lambda\neq-[\mu(n+1)+m+1]$, where $n,m\in\mathbb{N}$. $\delta^{(n)}$ is the nth derivate of the delta function. ∂_r^a is then defined by

$$(i)\quad \partial_r^a(\alpha f(r)+\beta g(r)) = \alpha\partial_r^a f(r)+\beta\partial_r^a g(r), \tag{B1}$$

$$(ii)\quad \partial^a r^e = \frac{\Gamma(e+1)}{\Gamma(e-a+1)}r^{e-a}+\sum_{n=0}^{\infty}\delta_{a,e+n+1}\Gamma(e+1)\delta^{(n)}(r), \tag{B2}$$

$$(iii)\quad \partial_r^a\delta^{(n)}(r) = \frac{r^{-1-n-a}}{\Gamma(-a-n)}+\sum_{m=0}^{\infty}\delta_{a,m}\delta^{(n+m)}(r). \tag{B3}$$

Here α and β are real constants and $g(r)$ is another function which can be expanded in the same way as $f(r)$. The most important properties of these fractional derivatives are

$$\partial_r^{a+b}f(r) = \partial_r^a\partial_r^b f(r) = \partial_r^b\partial_r^a f(r), \tag{B4}$$

$$[\partial_r^a,r]f(r) = (\partial_r r-r\partial_r)f(r) = a\partial_r^{a-1}f(r), \tag{B5}$$

$$\partial_r^a f(\alpha r) = \alpha^a\partial_{\alpha r}^a f(\alpha r), \tag{B6}$$

$$\partial_{\alpha r}^a f(r) = \alpha^{-a}\partial_r^a f(r). \tag{B7}$$

APPENDIX C: DERIVATION OF THE SCALING FUNCTION

In this appendix we outline the derivation of the scaling function $\phi(u)$ for any rational dynamical exponent z, thereby correcting the incomplete result given in [25]. For notational simplicity we do this in one space dimension. For $z=2$ and $z=4$ we give the results in two space dimensions at the end of this appendix.

Our starting point is Eq. (42):

$$(\partial_u+\hat{a}u\partial_u^{2-z}+\hat{b}\partial_u^{1-z})\phi(u) = 0 \tag{C1}$$

with $\hat{a}=z\lambda$ and $\hat{b}=2z(2-z)\gamma_1$. We write the rational dynamical exponent as $z=N+\frac{p}{q}$, where N is the largest integer equal or smaller than z. We also assume $\hat{a}\neq 0$ and give the result for $\hat{a}=0$ at the end, as it can be derived in exactly the same way. In a first step we rewrite (C1) as

$$(\partial_u^z+\hat{a}u\partial_u+\hat{b})\Psi(u) = 0 \tag{C2}$$

with $\Psi(u) = \partial_u^{1-z}\phi(u)$. In doing so we have used property (B4) of the fractional derivative. It is in fact this step which enables us to avoid the negative exponents for the fractional derivatives in Eq. (C1), which are responsible for the incomplete result in [25]. The solution $\Psi(u)$ of this equation yields then the scaling function $\phi(u)$ through the relation

$$\phi(u) = \partial_u^{z-1}\Psi(u). \tag{C3}$$

Before doing this, it is instructive to consider (C2) for the case $z=4$. Indeed, Eq. (C2) is then a normal differential equation of fourth order, so that the solution will be, *a priori*, a linear combination of four linearly independent solutions (i.e., it will contain four free parameters). The method applied in [25], however, only yields one of these linearly independent solutions.

We solve (C2) for $u>0$ under the additional assumption that the desired solution is nonsingular for $u \to 0$. Furthermore, we require that the scaling function should drop to zero for $u \to \infty$. We then make the ansatz

$$\Psi(u) = \sum_{n=0}^{\infty} c_n u^{n/q+s}, \quad c_0 \neq 0 \tag{C4}$$

and suppose $s > -1$ [60]. This ansatz is introduced into (C2) and yields, because of $c_0 \neq 0$, the recursion relation

$$c_{n+\xi}\frac{\Gamma((n+q)/q+s+1)}{\Gamma(n/q+s+1)} + c_n(\hat{a}(n/q+s)+\hat{b}) = 0 \tag{C5}$$

with $\xi = p + qN$ as well as the relation

$$s = \frac{p}{q} + m, \quad m \in \mathcal{E}, \tag{C6}$$

where the set $\mathcal{E}$ is given by

$$\mathcal{E} := \begin{cases} -1, 0, \dots, N-1, & p \neq 0, \\ 0, \dots, N-1, & p = 0. \end{cases} \tag{C7}$$

It then also follows that $c_1 = \dots = c_{p+qN-1} = 0$. With this we obtain after some algebra

$$\Psi(u) = \sum_{m \in \mathcal{E}} c_m \sum_{n=0}^{\infty} a_n^{(m)} u^{nz+p/q+m} \tag{C8}$$

with

$$a_n^{(m)} = \frac{(-\hat{a}z)^n \Gamma\left(\frac{p}{q}+1+m\right)\Gamma\left(n+\frac{p/q+m}{z}+\frac{\hat{b}/\hat{a}}{z}\right)}{\Gamma\left(nz+\frac{p}{q}+m+1\right)\Gamma\left(\frac{p/q+m}{z}+\frac{\hat{b}/\hat{a}}{z}\right)}. \tag{C9}$$

The c_m are free parameters not fixed by the theory. Finally, the scaling function $\phi(u)$ is obtained from Eq. (C3):

$$\phi(u) = \sum_{m \in \mathcal{E}} c_m \phi^{(m)}(u) \tag{C10}$$

with

$$\phi^{(m)}(u) = \sum_{n=0}^{\infty} b_n^{(m)} u^{(n-1)z+p/q+m+1} \tag{C11}$$

where the coefficients $b_n^{(m)}$ are given by

$$b_n^{(m)} = \frac{(-\hat{a}z)^n \Gamma\left(\frac{p}{q}+1+m\right)\Gamma\left(n+\frac{p/q+m}{z}+\frac{\hat{b}/\hat{a}}{z}\right)}{\Gamma\left((n-1)z+\frac{p}{q}+m+2\right)\Gamma\left(\frac{p/q+m}{z}+\frac{\hat{b}/\hat{a}}{z}\right)}. \tag{C12}$$

We remark that our final result (C10) is indeed regular for $u \to 0$, as $b_0^{(m)} = 0$ for $m = -1, \dots, N-2$. This is readily seen by recalling that $\Gamma(l) = \infty$ for $l \in -\mathbb{N}_0$. As already mentioned in [25], the radius of convergence is infinite for $z>1$.

We also note that the number of free parameters is *a priori* equal to N if $z \in \mathbb{N}$ and $N+1$ else. However, there might be cases where some of the independent solutions $\phi^{(m)}(u)$ vanish.

For completeness let us also quote the result for $\hat{a}=0$. In this case

$$b_n^{(m)} = (-\hat{b})^n \frac{\Gamma\left(\frac{p}{q}+1+m\right)\Gamma\left(nz+\frac{p}{q}+m+1\right)}{\Gamma((n+1)z+1)\Gamma\left((n-1)z+\frac{p}{q}+m+2\right)}. \tag{C13}$$

The expressions (43) and (44) for $\phi(u)$ can be obtaind from (C10) by setting $z=2$ (i.e., $p=0$ and $N=2$) or $z=4$ (i.e., $p=0$ and $N=4$), respectively. For the EW case it is important to note that $\phi^{(0)}(u)$ vanishes as $b_n^{(0)}=0$ for every n which immediately follows from the fact that $\hat{b}=2z(2-z)\gamma_1=0$ for $z=2$. The remaining solution $\phi^{(1)}(u)$ is then just the exponential function

$$\phi^{(1)}(u) = \exp(-\lambda u^2). \tag{C14}$$

For the MH case it is the solution $\phi^{(2)}$ which vanishes in the free-field case. Indeed, from Eq. (40) we obtain $\frac{\hat{b}}{4\hat{a}} = -\frac{1}{2}$ by recalling that $x=\frac{1}{2}$. It then follows that the Gamma-function $\Gamma\left(\frac{p/q+m}{z}+\frac{\hat{b}/\hat{a}}{z}\right)$ always diverges in the denominator of Eq. (C12), yielding $\phi^{(2)}(u)=0$ for every u. We are therefore left with three independent solutions, and after relabelling we obtain the final expression (44):

$$\phi(u) = \tilde{c}_0\left(-\frac{\lambda}{16}u^4\right)^{1/4} {}_0F_2\left(\frac{3}{4}, \frac{5}{4}, -\frac{\lambda}{16}u^4\right) + \tilde{c}_1\left(-\frac{\lambda}{16}u^4\right)^{1/2}$$

$$\times {}_0F_2\left(\frac{5}{4}, \frac{3}{2}, -\frac{\lambda}{16}u^4\right) + \tilde{c}_2 {}_0F_2\left(\frac{1}{2}, \frac{3}{4}, -\frac{\lambda}{16}u^4\right). \tag{C15}$$

Let us add that the asymptotic behavior of the generalized hypergeometric functions $_0F_2$ is well known [46,47]. Recalling that the scaling function should vanish for $u \to \infty$, we can exploit this known asymptotic behavior in order to derive relations between the parameters c_m. For the case $z=4$ this then yields the condition (45) given in Sec. IV.

Let us finish this appendix by quoting the resulting scaling functions $\phi(u)$ in two space dimensions. For the EW case with $z=2$ we get the same expression (C14) as for the one-dimensional case. For the MH case with $z=4$, our calculations yield the expression

$$\phi(u) = \widetilde{c}_1 \left(-\frac{\lambda}{16}u^4\right)^{1/2} {}_0F_2\left(\frac{3}{2}, \frac{3}{2}, -\frac{\lambda}{16}u^4\right)$$
$$+ \widetilde{c}_2 \, {}_0F_2\left(\frac{1}{2}, 1, -\frac{\lambda}{16}u^4\right) \tag{C16}$$

with the additional condition

$$\widetilde{c}_1 = -\frac{4}{\sqrt{\pi}}\widetilde{c}_2. \tag{C17}$$

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